Strange nonchaotic attractors in autonomous and periodically driven systems

V. S. Anishchenko, T. E. Vadivasova, and O. Sosnovtseva
Physics Department, Saratov State University, Astrakhanskaya strasse 83, Saratov 410071, Russia

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We demonstrate that a strange nonchaotic attractor can be realized not only in quasiperiodically driven systems but also in autonomous and periodically forced systems. We show that the destruction of an ergodic torus via a band-merging crisis and the appearance of a strange nonchaotic attractor are applicable to a wide class of dynamical systems. [S1063-651X(96)07109-7]

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I. INTRODUCTION

A strange nonchaotic attractor (SNA) is one of the nontrivial attracting sets that have been found and investigated over the last years [1–6]. A SNA is strange in a geometrical sense; i.e., it has a fractal structure. On the other hand, the exponential divergence of the trajectories is absent from a SNA; i.e., it is not chaotic.

The properties of strange nonchaotic attractors have previously been investigated in a variety of systems, but in all cases the SNA was found when a quasiperiodic forcing was applied. Such systems are characterized by the presence of at least two incommensurate frequencies with an irrational ratio. This condition creates a rough ergodic torus, which can be destroyed when parameters are varied. The band-merging crisis of an n-band ergodic torus has been suggested as one mechanism for generating a SNA [7]. The crisis is related to the homoclinic touching of manifolds of a saddle torus coexisting with a stable torus in the phase space of the system. This mechanism is not unique. The gradual fractalization of a two-dimensional ergodic torus also apparently leads to a SNA [8].

Given that the appearance of a SNA is related to the destruction of a two-dimensional ergodic torus, we are faced with the following question: Is the SNA regime typical only of quasiperiodically driven systems or can it be observed in other systems as well?

It is known that the regions of ergodic quasiperiodic motion in the parameter space of autonomous systems in $\mathbb{R}^3$ and periodically driven systems in $\mathbb{R}^3$ can have nonzero measure. Hence, a rough two-dimensional ergodic torus T^2 can be realized in these systems. When the nonlinearity of the system is increased, the measure of the quasiperiodic region is generally decreased while the measure of synchronization regions is increased. Synchronization is thus associated with the destruction ergodic tori in three-dimensional systems. However, n-band two-dimensional tori (nT^2) in the region of quasiperiodic motion can be observed in systems with a phase space of dimension $N \geqslant 4$. This suggests that a SNA can appear via a band-merging crisis in these systems as well.

The goals of this paper are to reveal band-merging bifurcations of ergodic tori in systems without external quasiperiodic forcing and to verify that the attractor arising after the band-merging crisis is a SNA.

The paper is organized as follows. In Sec. II we describe the basic properties of two coupled ring maps and a periodically forced oscillator with an inertial nonlinearity. In Sec. III we show that a SNA exists in certain parameter ranges of the autonomous system of two coupled ring maps. The SNA is characterized by computation of autocorrelation functions. In Sec. IV we analyze the appearance of a SNA in a system of differential equations describing the dynamics of an oscillator with an inertial nonlinearity. We summarize the results in Sec. V.

II. BASIC MODELS

We investigate two models: an autonomous map in $\mathbb{R}^4$ and a differential system with harmonic forcing in $\mathbb{R}^3$.

The first model is two asymmetrically coupled ring maps:

$$\begin{aligned} x_{n+1} &= x_n + \Omega_1 - (K_1/2\pi)\sin(2\pi x_n) + \gamma_1 y_n \\ &\quad + A\cos(2\pi u_n), \quad \text{mod } 1 \\ y_{n+1} &= \gamma_1 y_n - (K_1/2\pi)\sin(2\pi x_n), \\ u_{n+1} &= u_n + \Omega_2 - (K_2/2\pi)\sin(2\pi u_n) + \gamma_2(y_n + v_n), \\ &\quad \text{mod } 1 \\ v_{n+1} &= \gamma_2(y_n + v_n) - (K_2/2\pi)\sin(2\pi u_n). \end{aligned} \tag{1}$$

In the case of $\gamma_2 = 0$, the system (1) is the ring map with the quasiperiodic forcing at some values of the parameters Ω_2, K_2. The winding number of this map is determined by the parameters Ω_2 and K_2 and can be equal to an irrational value (the reciprocal of the golden mean, for example).

When a small coupling coefficient ($\gamma_2 \neq 0$) is introduced, feedback appears between the ring maps, and the analogy between system (1) and the quasiperiodically driven map is broken down. In this case, the system is two coupled maps.

The second model is a periodically driven oscillator with an inertial nonlinearity. The dynamical equations of the system in dimensionless variables have the following form:

$$\begin{aligned} \dot{x} &= mx + y - xz + A\sin(p\tau), \\ \dot{y} &= -x, \\ \dot{z} &= -gz + gf(x), \end{aligned} \tag{2}$$

where

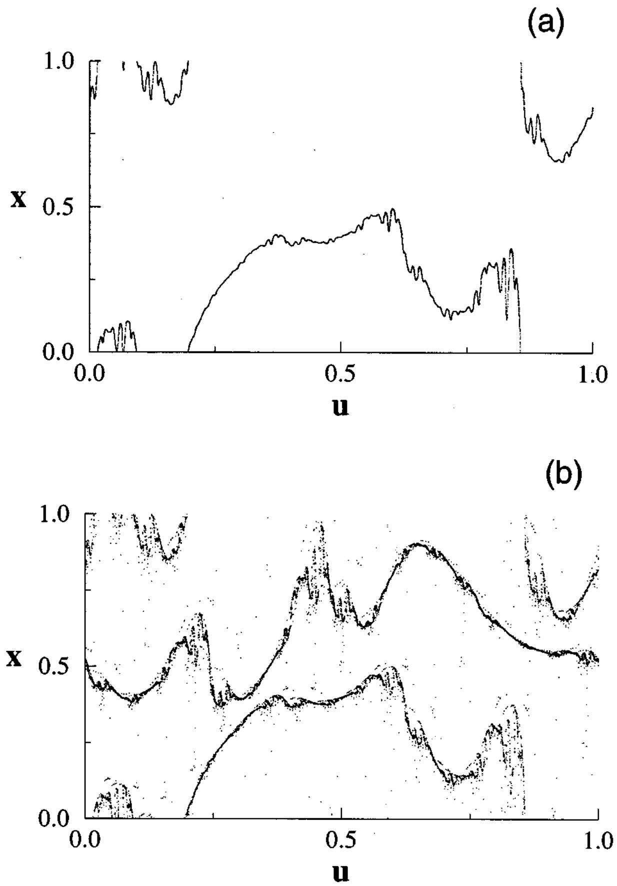

FIG. 1. Phase portraits of nonstrange and strange nonchaotic attractors for system (1) at $K_1=0.8783$ (a) and $K_1=0.8784$ (b). Points are plotted for every second iteration. The largest Lyapunov exponents for these cares are $\lambda_1=-0.658\,35\times10^{-4}$ (a) and $\lambda_1=0.188\,33\times10^{-3}$ (b). These exponents are equal to zero within the limits of numerical accuracy.

$$f(x)=\begin{cases} x^2, & x\geqslant 0 \\ 0, & x<0. \end{cases}$$

The parameters m and g determine the natural dynamics of system (2). The parameter A is the forcing amplitude, p is the normalized forcing frequency, and τ is the dimensionless time. The dynamics of system (2) was studied in detail in [9].

III. BAND-MERGING CRISIS IN COUPLED RING MAPS

As mentioned above, system (1) can be considered as a quasiperiodically driven ring map when $\gamma_2=0$. In this case there are regimes of two-band ergodic tori $2T^2$ (in terms of a map, they correspond to a closed invariant curve) at some parameter values. The band-merging crisis leads to the appearance of a SNA, which exists at a limit range of the parameter space. We focus on the question of whether this phenomenon takes place when a small coupling ($\gamma_2\neq 0$) is introduced.

The parameter K_1 is varied while the other parameters are fixed ($\Omega_1=0.5$, $\Omega_2=0.5(\sqrt{5}-1)$, $K_2=0.03$, $A=0.4$, $\gamma_1=\gamma_2=0.01$). In the case $\gamma_2\neq 0$, the bifurcation associated with the band merging of the invariant curve is found in the region of ergodic motion at $K_1=K_1^*$ ($0.8783<K_1^*<0.8784$). Figures 1(a) and 1(b) plotted for each second iteration show the xu projection of phase trajectories before and after the band-merging crisis, respectively. In Fig. 1(a) only one band of the invariant curve can be observed, while Fig. 1(b) shows the merging of two bands. It is important to note that the Lyapunov exponents are not sensitive to this bifurcation (Fig. 2). Chaotic dynamics corresponding to $\lambda_1>0$ occurs at larger values of the parameter K_1 ($K_1>1.1$). Negative values of λ_1 in some ranges of K_1 are due to the synchronization that inevitably occurs when coupling is introduced.

The band-merging bifurcation leads to a SNA. While revealed in numerical simulations, there is as yet no theorem

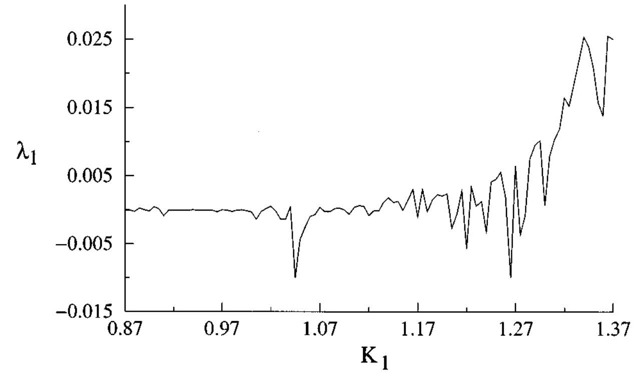

FIG. 2. The largest Lyapunov exponent vs the parameter K_1.

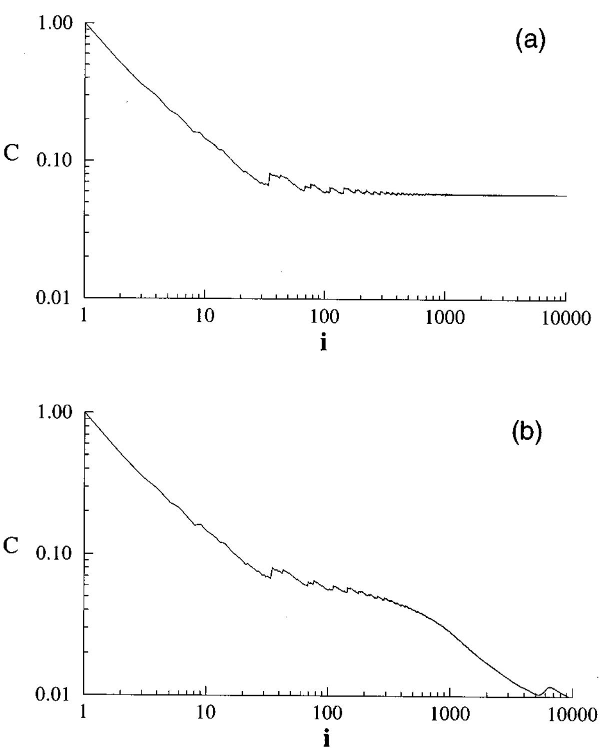

FIG. 3. The averaged squared autocorrelation function for the variable x for the case of nonstrange (a) and strange (b) nonchaotic attractors. Note the logarithmic scale of the axes.

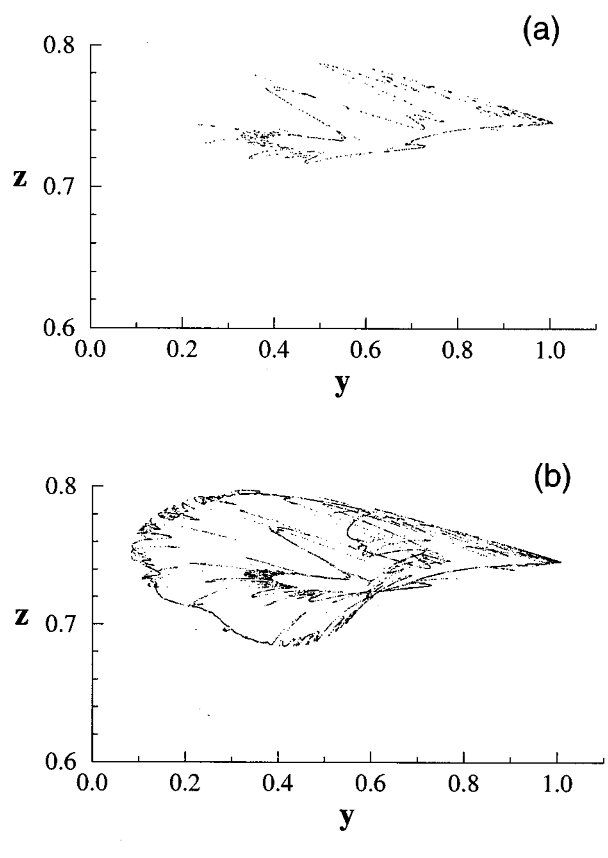

FIG. 4. Poincaré sections of nonstrange (a) and strange (b) nonchaotic attractors for system (2), plotted for every fourth iteration. The largest Lyapunov exponents for these cases are $\lambda_1 = -0.371\times10^{-3}$ (a) and $\lambda_1 = -0.265\times10^{-3}$ (b). These exponents are equal to zero within the limits of numerical accuracy.

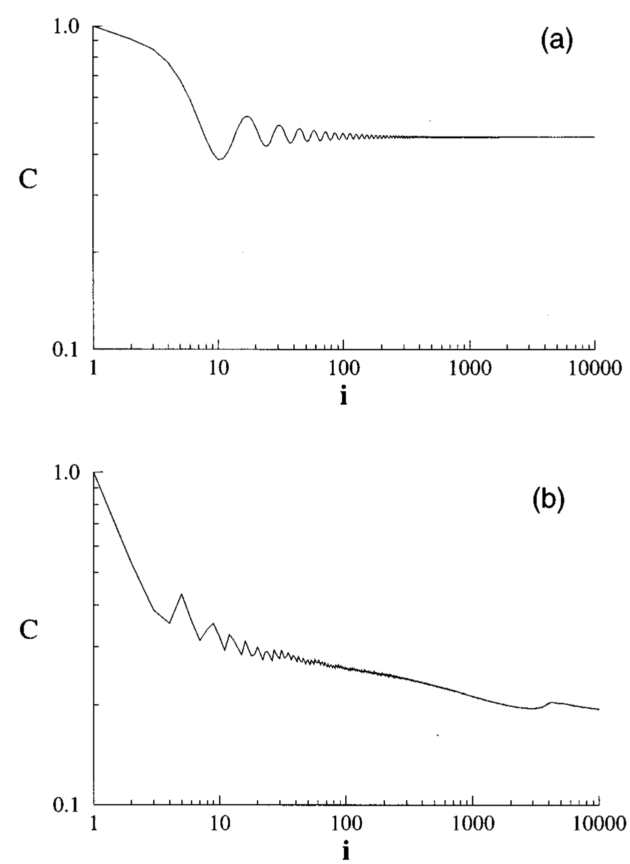

FIG. 5. The averaged squared autocorrelation function for the variable y for the regime of nonstrange (a) and strange (b) nonchaotic attractors.

proving the existence of a SNA in this instance. Therefore, it is necessary to confirm the fractal structure of the attractor. It was shown in [10,11] that a SNA has a singular continuous spectrum. To diagnose this kind of spectrum we use the method proposed in [12]. The autocorrelation function $\psi_x(i)$ is determined as follows:

$$\psi_x(i) = \frac{\langle x(n)x(n+i)\rangle - \langle x(n)\rangle\langle x(n+i)\rangle}{\langle x^2(n)\rangle - \langle x(n)\rangle^2},$$

where $n=0,1,2,\ldots$ is a discrete time coordinate, and $i=0,1,2,\ldots$ is a time shift.

To represent a singular continuous spectrum, $\psi_x(i)$ must satisfy the conditions

$$\lim_{i\to\infty} \psi_x(i) \neq 0,$$

$$\lim_{i\to\infty} C(i) = 0,$$

where

$$C(i) = \frac{1}{i}\sum_{j=1}^{i-1} \psi_x^2(j).$$

The functions $\psi_x(i)$, $C(i)$ for $x(n)$ have been calculated for both cases: before the band merging ($K_1=0.8783$) and after it ($K_1=0.8784$). The first condition means that the spectrum is not continuous, and is met in both cases because the band-merging bifurcation does not lead to chaos. Figures 3(a) and 3(b) show that the average squared autocorrelation function $C(i)$ goes to some constant level before the bifurcation ($K_1=0.8783$) while the character of this function is changed after the bifurcation ($K_1=0.8784$). This behavior of $C(i)$ confirms the fact that the attractor arising at the band merging has a singular continuous spectrum and is therefore a SNA. Calculating $C(i)$ versus the parameter K_1, we have found that the SNA does not exist everywhere for $K_1 > K_1^*$ but is found within narrow parameter intervals, which alternate with the regions of synchronization.

IV. BAND-MERGING CRISIS IN SELF-OSCILLATOR WITH PERIODICAL FORCING

It was shown in [9] that there is a four-band two-dimensional torus $4T^2$ in the phase space of system (2). When parameters are varied, the $4T^2$ torus is destroyed and a transition to chaos takes place. However, we have found that the $4T^2$ torus undergoes a band-merging bifurcation before the chaotic attractor appears. The excitation parameter m is chosen as a control parameter while the other parameters are fixed ($g=0.3, A=0.3, p=0.111$). The band-merging bifurcation of the ergodic $4T^2$ torus occurs at $m=1.0621$. As a result of this bifurcation, a two-band attractor appears, but it is not chaotic because the largest Lyapunov exponent tends to zero within the limits of numerical accuracy. The signature of the spectrum of the Lyapunov exponents remains the same (0, ''0,'' ''$-$,'' ''$-$'') before and after the bifurcation. The projections of Poincaré sections of the attractors by the secant surface $x=0$ at $m=0.062\,05$ and $m=0.0621$ are shown in Figs. 4(a) and 4(b), respectively. Plotting each fourth iteration of the Poincaré section, Fig. 4(a) presents one band of the four-band invariant curve, while the band merging can be distinguished in Fig. 4(b).

To detect the existence of the SNA, the behavior of the function $C(i)$ was investigated. The autocorrelation function was calculated for the sequence of points $y(n)$ of Poincaré section; each second point was taken to avoid a periodic component after the band merging. The calculation of the autocorrelation function for system (2) relies on the fact that the return time for the secant surface is constant in the regime being considered. Figures 5(a) and 5(b) display the function $C(i)$ obtained for the attractor before and after the bifurcation, respectively. Although the function $C(i)$ decreases more slowly than in system (1), the character of the function $C(i)$ after the bifurcation is the same. Thus, in system (2) a SNA emerges as a result of the band-merging crisis of an ergodic two-dimensional torus.

V. CONCLUSIONS

We have found numerically that a band-merging crisis of the two-dimensional tori can lead to a SNA not only for the systems with quasiperiodic forcing but also for the systems with periodic excitation and systems of coupled oscillators. However, open questions remain concerning whether other mechanisms of SNA appearance are possible in such systems and whether there is a transition from SNA to chaos excluding the synchronization phenomenon on the torus. It is clear that SNA regimes can be realized in a wide class of dynamical systems and are not restricted to systems with quasiperiodic forcing.

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